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Longest path partitions in generalizations of tournaments

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Abstract

We consider the so-called Path Partition Conjecture for digraphs which states that for every digraph, D, and every choice of positive integers, λ_1, λ_2, such that $\lambda_1 + \lambda_2$ equals the order of a longest directed path in D, there exists a partition of D into two digraphs, D_1 and D_2, such that the order of a longest path in D_i is at most λ_i, for $i = 1, 2$.

We prove that certain classes of digraphs, which are generalizations of tournaments, satisfy the Path Partition Conjecture and that some of the classes even satisfy the conjecture with equality.

Keywords: Longest path; Path partition conjecture; Locally in-semicomplete digraph; Quasi-transitive digraph; Extended semicomplete digraph

1. Introduction

Given a (di)graph, D, $\lambda(D)$ denotes the order of a longest (directed) path in D. An *independent set* in D is a subset, $S \subseteq V(D)$, of the vertices of D, such that no two vertices of S are adjacent in D. The Gallai–Roy–Vitaver Theorem (see e.g. [3, Theorem 8.4.1]) states that the chromatic number of (the underlying graph of) a digraph D is at most $\lambda(D)$. In 1983 Laborde, Payan, and Xuong posed the following conjecture which extends this theorem in a natural way.

Conjecture 1 (*Laborde et al. [14]*). *Every digraph, D, contains an independent set, X, such that $\lambda(D - X) < \lambda(D)$.*

The corresponding statement for undirected graphs is easily seen to be true (just take any maximal independent set). Conjecture 1 seems very difficult, however, and only a few partial results have been obtained. Clearly, if the digraph has a kernel, then removing any kernel will decrease the order of every longest path. Havet [13] verified Conjecture 1 for digraphs with independence number at most two.

The following, more general, conjecture is known as the *Path Partition Conjecture*:

Conjecture 2 (*Laborde et al. [14]*). *For every digraph, D, and every choice of positive integers, λ_1, λ_2, such that $\lambda(D) = \lambda_1 + \lambda_2$, there exists a partition of D into two digraphs, D_1 and D_2, such that $\lambda(D_i) \leqslant \lambda_i$, for $i = 1, 2$.*

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The undirected version of Conjecture 2 is also called the Path Partition Conjecture. Some partial results on that conjecture have been obtained in [7–9]. Very few results are known on the directed version (see e.g. [1]).

A seemingly stronger version of the conjecture is stated in [6]. Bondy attributes it to Laborde et al. [14] although only the undirected version of Conjecture 2 is explicitly mentioned there.

Conjecture 3 (*Bondy [6]*). *For every digraph, D, and every choice of positive integers, λ_1, λ_2, such that $\lambda(D)=\lambda_1+\lambda_2$, there exists a partition of D into two digraphs, D_1 and D_2, such that $\lambda(D_i)=\lambda_i$, for $i=1,2$.*

All three conjectures above are very difficult to attack for general digraphs, since very little can be said about the structure of longest paths in general digraphs. To give an indication of the way structure can simplify the conjectures, let us consider a few examples. A digraph is *semicomplete* if it has no pair of non-adjacent vertices. A digraph is *locally semicomplete* if the in-neighbours of every vertex induce a semicomplete digraph and the out-neighbours of every vertex induce a semicomplete digraph. All three conjectures are trivial for connected locally semicomplete digraphs (and hence for semicomplete digraphs) as every such digraph has a hamiltonian path [2]. The proof of Conjecture 2 is even more trivial in the case of bipartite digraphs, since we may simply let D_1 and D_2 be the arc-less digraphs induced by the two independent sets of an arbitrary bipartition.

In this paper we will mainly consider digraphs that are generalizations of tournaments. Here the structure of longest paths is often quite well understood (see e.g. [3]).

A digraph is *quasi-transitive* if the presence of arcs xy and yz implies an arc between x and z (if we require the arc to go from x to z then D is *transitive*). Thus, quasi-transitive digraphs generalize semicomplete digraphs which, in turn, generalize tournaments. Quasi-transitive digraphs were introduced in [4] and by now a lot is known about them, see e.g. [3]. In particular, they have a certain recursive structure (see Theorem 4) which enables the development of efficient (polynomial) algorithms for many problems which are $\mathscr{NP}$-hard for general digraphs (see e.g. [5,11]).

We shall prove (an extension of) Conjecture 2 for quasi-transitive, extended semicomplete, and locally in-semicomplete digraphs (the latter two classes will be defined below) and Conjecture 3 for locally in-semicomplete and extended semicomplete digraphs.

2. Terminology and preliminaries

Terminology not defined below is consistent with [3].

For a digraph, $D=(V,A)$, the *order* (*size*) of D is the cardinality of $V(A)$. We will denote the order (size) of a digraph under consideration by n (m). An arc from x to y in D will be denoted by xy or $x\to y$ and we say that x *dominates* y.

The *underlying* graph, $UG(D)$, of a digraph, D, is the undirected graph with the same vertices as D and which has an edge, xy, for each pair, $x,y\in V(D)$, such that $x\to y$ or $y\to x$ (or both). The complement, $\overline{G}$, of an undirected graph, G, is the graph with vertex set $V(G)$ in which two vertices are adjacent if and only if they are not adjacent in G.

Paths and cycles will always be directed. An *oriented graph* is a digraph without cycles of length two.

Two vertices, x,y, in a digraph, D, are *similar* if they have the same in-neighbours and the same out-neighbours. For a digraph, $D=(V,A)$, and a set, $X\subseteq V$, $D\langle X\rangle$ is the subdigraph induced by X. For a pair of distinct vertices, x,y, on a cycle, C, $C[x,y]$ is the subpath of C from x to y.

A *k-path-q-cycle subdigraph* (*k-path-q-cycle factor*), $\mathscr{F}$, of a digraph, D, is a (spanning) collection of k paths and q cycles, all disjoint. When $k=0$, $\mathscr{F}$ is a *q-cycle subdigraph* (*and a q-cycle factor* if it is spanning) and when $q=0$, $\mathscr{F}$ is a *k-path subdigraph* (and a *k-path factor* if it is spanning). A k-path-q-cycle subdigraph in which q may be arbitrary (including zero) is called a *k-path-cycle subdigraph*.

For a given digraph, D, let $\lambda_k(D)$ denote the maximum number of vertices contained in a k-path subdigraph of D. A k-path subdigraph of D which covers $\lambda_k(D)$ vertices is called a *maximum* k-path subdigraph of D. Note that $\lambda_1(D)=\lambda(D)$. We say that a digraph D is *traceable* if D has a hamiltonian path, i.e. $\lambda(D)=|V(D)|$.

For a digraph, R, with vertex set $V(R)=\{u_1,u_2,\dots,u_r\}$, and digraphs, $H_1,H_2,\dots,H_r$, let $D=R[H_1,H_2,\dots,H_r]$ be the digraph with vertex set $V(D)=V(H_1)\cup\cdots\cup V(H_r)$ in which $xy\in A(D)$ if and only if $x\in V(H_i)$, $y\in V(H_j)$ and $u_iu_j\in A(R)$, where $i\neq j$, or $i=j$ and $xy\in A(H_i)$. In other words, D is obtained from R by substituting the digraph H_i for vertex u_i, for each $i=1,2,\dots,r$.

D is an *extended semicomplete digraph* if there is a semicomplete digraph, S, and independent sets, $E_1, \ldots, E_s$, such that $D = S[E_1, \ldots, E_s]$. A digraph is *locally in-semicomplete* if the in-neighbours of every vertex induce a semicomplete digraph.

3. Auxiliary results

We shall make use of several results on generalizations of tournaments.

Theorem 4 (*Bang-Jensen and Huang [4]*). *Let D be a quasi-transitive digraph.*

(a) *If D is not strong then $D = T[H_1, H_2, \ldots, H_t]$ for some transitive oriented graph, T, where the H_i are the strong components of D.*
(b) *If D is strong then $D = S[Q_1, Q_2, \ldots, Q_s]$ for some strong semicomplete digraph, S, where the Q_i are the subdigraphs of D such that $\overline{UG(Q_i)}$ are the connected components of $\overline{UG(D)}$. Each Q_i is either a non-strong quasi-transitive digraph or a single vertex and if $q_i \to q_j \to q_i$ is a 2-cycle in S then both of Q_i and Q_j is a single vertex.*

Note that the above decomposition of a quasi-transitive digraph is unique as the (strongly) connected components of a (di)graph are unique.

We shall make extensive use of the following structural characterization of longest cycles in extended semicomplete digraphs.

Theorem 5 (*Bang-Jensen et al. [5]*). *Let $D = S[E_1, E_2, \ldots, E_s]$ be a strong extended semicomplete digraph. For $i = 1, 2, \ldots, s$, let m_i denote the maximum number of vertices from E_i which can be covered by a cycle of D. Then every longest cycle of D contains precisely m_i vertices from E_i, $i = 1, 2, \ldots, s$.*

Theorem 6 (*Bang-Jensen and Gutin [3, Theorems 2.7.7 and 3.11.11]*). *In $\mathcal{O}(|V(D)|^3)$ we can find a longest cycle in an extended semicomplete digraph, D.*

Lemma 7 (*Gutin [10]*). *Let D be an extended semicomplete digraph with a path, P, and a cycle, C, disjoint from P. There exists a path, P', of D with $V(P') = V(P) \cup V(C)$. Furthermore, given P and C, one can construct P' in time $\mathcal{O}(|V(P)| \cdot |V(C)|)$.*

4. Main results

4.1. Proof of Conjecture 2 for quasi-transitive, extended semicomplete, and locally in-semicomplete digraphs

Lemma 8. *Let $D = S[E_1, E_2, \ldots, E_s]$ be an extended semicomplete digraph and let $l_{i,k}$ denote the maximum number of vertices of E_i that can be covered by a k-path subdigraph in D. Then every maximum k-path subdigraph in D covers exactly $l_{i,k}$ vertices of E_i, for every $i = 1, 2, \ldots, s$.*

Proof. Let D^* be the extended semicomplete digraph obtained from D by adding k new independent vertices, $E_{s+1} = \{s_1, s_2, \ldots, s_k\}$, and joining each of them to every vertex of D by an arc in both directions (i.e., forming a 2-cycle with every vertex of D). Clearly, D^* is a strong extended semicomplete digraph and the maximum number of vertices of E_i which lie on a cycle in D^* is $l_{i,k}$, for $i = 1, 2, \ldots, s$, and k, for $i = s + 1$. Now the claim follows by applying Theorem 5 to D^*. $\square$

Lemma 9. *Let $D = S[Q_1, Q_2, \ldots, Q_s]$, where S is a strong semicomplete digraph and each Q_i is either a single vertex or a non-strong quasi-transitive digraph. For every $k \in \{1, 2, \ldots, |V(D)|\}$ and $i \in \{1, 2, \ldots, s\}$, there exists* *an integer, $v_{i,k}$, such that every maximum k-path subdigraph, P_k, of D satisfies $|V(Q_i) \cap V(P_k)| = v_{i,k}$ and no k-path subdigraph of D contains more than $v_{i,k}$ vertices of Q_i.*

Proof. Let $D' = S[E_1, E_2, \ldots, E_s]$, where E_i is a set of $|V(Q_i)|$ independent vertices, $i = 1, 2, \ldots, s$. Clearly, D' is a strong extended semicomplete digraph. Define $l_{i,k}$ as in Lemma 8. Let $v_{i,k}$ be the maximum number of vertices in an $l_{i,k}$-path subdigraph in Q_i. By Lemma 8, there exists a k-path subdigraph in D' containing exactly $l_{i,k}$ vertices from E_i, $i = 1, 2, \ldots, s$. This k-path subdigraph can be extended to a k-path subdigraph in D containing $v_{i,k}$ vertices from Q_i, by substituting each of the $l_{i,k}$ vertices of E_i by a path from an $l_{i,k}$-path subdigraph ofQ_i which covers $v_{i,k}$ vertices. Furthermore, it is not difficult to see that no k-path subdigraph in D can include more than $v_{i,k}$ vertices from Q_i, as this would imply that E_i could be visited more than $l_{i,k}$ times in D'. Therefore, every maximum k-path subdigraph, P_k, of D satisfies $|V(Q_i) \cap V(P_k)| = v_{i,k}$, $i = 1, 2, \ldots, s$. $\square$

Theorem 10. *Let D be a quasi-transitive digraph or a strong extended semicomplete digraph, and let q be any positive integer. Then there exists a partition, (A, B), of $V(D)$ such that the following holds:*

(i) $\lambda(D\langle A\rangle) \leqslant q$;
(ii) $\lambda_k(D\langle B\rangle) \leqslant \lambda_k(D) - q$ *for all* $k = 1, 2, 3, \ldots, |V(B)|$, *provided* $\lambda_k(D) - q \geqslant 0$.

Proof. We shall prove the theorem by induction on $|V(D)|$. For the base case, when $|V(D)| = 1$, the claim is trivially true.

Suppose first that D is strong. By Theorem 4 or the definition of extended semicomplete, we may let $D = S[Q_1, Q_2, \ldots, Q_s]$, where S is a strong semicomplete digraph and each Q_i is either a non-strong quasi-transitive digraph (in particular, it could be an independent set of vertices) or a single vertex, for $i = 1, 2, \ldots, s$. Let $v_{i,k}$ be defined as in Lemma 9 and assume without loss of generality that $v_{i,1} < |Q_i|$, for $i \in \{1, 2, \ldots, l\}$ and $v_{j,1} = |Q_j|$, for $j \in \{l+1, l+2, \ldots, s\}$ (i.e., there is no path in D containing all the vertices of Q_i, for any $i \in \{1, 2, \ldots, l\}$, but there is a path in D containing all the vertices of $Q_{l+1} \cup Q_{l+2} \cup \cdots \cup Q_s$). We can assume that $q < v_{1,1} + v_{2,1} + \cdots + v_{s,1}$, since otherwise $(A, B) = (V(D), \emptyset)$ is the desired partition. Now define r and γ_r such that the following holds:

$$\gamma_r = v_{1,1} + v_{2,1} + \cdots + v_{r-1,1} < q \leqslant v_{1,1} + v_{2,1} + \cdots + v_{r,1}.$$

We now consider the cases $r \leqslant l$ and $r > l$ separately.

$r \leqslant l$: Let $q' = q - \gamma_r > 0$ and use our induction hypothesis to partition Q_r into (A_r, B_r) such that $\lambda(Q_r\langle A_r\rangle) \leqslant q'$ and $\lambda_k(Q_r\langle B_r\rangle) \leqslant \lambda_k(Q_r) - q'$, for all $k = 1, 2, 3, \ldots, |V(B_r)|$. Let $A = V(Q_1) \cup V(Q_2) \cup \cdots \cup V(Q_{r-1}) \cup A_r$ and let $B = B_r \cup V(Q_{r+1}) \cup V(Q_{r+2}) \cup \cdots \cup V(Q_s)$. We will now show that A and B fulfill the conditions (i) and (ii) in the theorem.

Denote the vertices of S by $V(S) = \{1, 2, \ldots, s\}$, where i has been expanded to Q_i in D. We first show that $S_l = S\langle\{1, 2, \ldots, l\}\rangle$ is acyclic. Indeed, assume that $C = c_1c_2 \ldots c_zc_1$ is a cycle in S_l, and without loss of generality assume that $|V(Q_{c_1})| \leqslant |V(Q_{c_j})|$ for all $j = 2, 3, \ldots, z$. It is not difficult to see (by going around the cycle, C, $|V(Q_{c_1})|$ times) that there exists a cycle (and, hence, a path) in D which contains all the vertices in Q_{c_1}, and hence it follows from Lemma 9 that $v_{c_1,1} = |Q_{c_1}|$, which contradicts the definition of l. Therefore S_l is acyclic and hence, if $i \leqslant l$, no path in D visits Q_i more than once. This implies the following:

$$\begin{aligned}\lambda(D\langle A\rangle) &= \lambda(Q_1) + \lambda(Q_2) + \cdots + \lambda(Q_{r-1}) + \lambda(Q_r\langle A_r\rangle) \\ &\leqslant \gamma_r + q' \\ &= q,\end{aligned}$$

where the first equality follows from the fact that $S\langle\{1, 2, \ldots, r\}\rangle$ is semicomplete and hence traceable. Let $k \in \{1, 2, 3, \ldots, |V(B)|\}$ be arbitrary, let W_k be a maximum k-path subdigraph of $D\langle B\rangle$, and assume that $W_k \cap Q_r$ consists of $b \geqslant 0$ paths. Then the intersection of Q_r with any maximum k-path subdigraph of D consists of at least b paths, and hence $v_{r,k} \geqslant \lambda_b(Q_r)$ (where we define $\lambda_0(\cdot) = 0$). If $b > 0$, we may assume that $q' < \lambda(Q_r)$, which implies that

$\lambda_b(Q_r) - q' > 0$; hence, by our induction hypothesis, $\lambda_b(Q_r\langle B_r\rangle) \leqslant \lambda_b(Q_r) - q' \leqslant v_{r,k} - q'$. Thus, for any value of b, we have:

$$\begin{aligned}\lambda_k(D\langle B\rangle) &\leqslant \lambda_b(Q_r\langle B_r\rangle) + v_{r+1,k} + v_{r+2,k} + \cdots + v_{s,k}\\ &\leqslant v_{r,k} + v_{r+1,k} + \cdots + v_{s,k} - q'\\ &= v_{1,1} + v_{2,1} + \cdots + v_{r-1,1} + v_{r,k} + v_{r+1,k} + \cdots + v_{s,k} - q\\ &\leqslant v_{1,k} + v_{2,k} + \cdots + v_{r-1,k} + v_{r,k} + v_{r+1,k} + \cdots + v_{s,k} - q\\ &= \lambda_k(D) - q.\end{aligned}$$

$r > l$: Let $q' = q - \gamma_r > 0$ and let A_r be any subset of q' vertices from Q_r. Let $B_r = V(Q_r) - A_r$, and define A and B as we did in the previous case. By Lemma 9:

$$\begin{aligned}\lambda(D\langle A\rangle) &\leqslant v_{1,1} + v_{2,1} + \cdots + v_{r-1,1} + |A_r|\\ &= \gamma_r + (q - \gamma_r).\end{aligned}$$

Therefore (i) holds; we will now prove (ii). Let $k \in \{1, 2, 3, \ldots, |V(B)|\}$ be arbitrary and note that $v_{r,k} = |Q_r|$. This implies the following:

$$\begin{aligned}\lambda_k(D\langle B\rangle) &\leqslant |B_r| + v_{r+1,k} + v_{r+2,k} + \cdots + v_{s,k}\\ &= (v_{r,k} - q') + v_{r+1,k} + \cdots + v_{s,k}\\ &= v_{1,1} + v_{2,1} + \cdots + v_{r-1,1} + v_{r,k} + v_{r+1,k} + \cdots + v_{s,k} - q\\ &\leqslant v_{1,k} + v_{2,k} + \cdots + v_{r-1,k} + v_{r,k} + v_{r+1,k} + \ldots + v_{s,k} - q\\ &= \lambda_k(D) - q,\end{aligned}$$

which completes the case when D is strong.

Now suppose D is a non-strong quasi-transitive digraph. By Theorem 4, there is a transitive oriented graph, T, and strong quasi-transitive digraphs, H_i ($i = 1, 2, \ldots, t$), such that $D = T[H_1, H_2, \ldots, H_t]$. Define p_i^{in} as the maximum number of vertices on a path in $D - V(H_i)$, such that the terminal vertex on the path has an arc into H_i. Define p_i^{end} to be the maximum number of vertices on a path in D, such that the terminal vertex on the path belongs to H_i. We will place vertices of D into A and B as follows:

(a) If $p_i^{\text{end}} \leqslant q$: Put $V(H_i)$ into A.
(b) If $p_i^{\text{in}} \geqslant q$: Put $V(H_i)$ into B.
(c) If $p_i^{\text{in}} < q < p_i^{\text{end}}$: Let $q_i' = q - p_i^{\text{in}}$ and use our induction hypothesis to partition H_i into (A_i, B_i) such that $\lambda(H_i\langle A_i\rangle) \leqslant q_i'$ and $\lambda_k(H_i\langle B_i\rangle) \leqslant \lambda_k(H_i) - q_i'$, for all $k = 1, 2, 3, \ldots, |V(B_i)|$. Put A_i into A and put B_i into B.

The above defines our partition, so we will now show that (i) and (ii) hold. Let P be a longest path in $D\langle A\rangle$, and assume that the terminal vertex belongs to H_i. If H_i was considered in (a) then clearly $|V(P)| \leqslant p_i^{\text{end}} \leqslant q$ and if H_i was considered in (c) then $|V(P)| \leqslant p_i^{\text{in}} + q_i' = q$ (since, for every j such that $H_i \rightarrow H_j$, we have $H_j \subseteq B$).

Let $k \in \{1, 2, \ldots, |V(B)|\}$ be arbitrary and let W_k be a k-path subdigraph of $D\langle B\rangle$, such that $|V(W_k)| = \lambda_k(D\langle B\rangle)$. Let P be any path in W_k and assume that P starts in the vertex $x \in V(H_i)$. If H_i was considered in (b), then there is a path P' in D, such that $|V(P')| \geqslant q$ and the terminal vertex in P' dominates x. However, merging P and P' into one path and considering this path together with the $k - 1$ paths in $W_k - P$, we see that $\lambda_k(D) \geqslant \lambda_k(D\langle B\rangle) + q$. Therefore we may assume that H_i was considered in (c).

Suppose that b paths of W_k start in H_i. By our induction hypothesis we know that $\lambda_b(H_i\langle B_i\rangle) \leqslant \lambda_b(H_i) - q_i'$. We can obtain a k-path subdigraph of D by substituting the b paths in $H_i\langle B_i\rangle$ by b paths in H_i and prepending a path of order

p_i^{in} to one of the paths (as we did above). This implies that:

$$\begin{aligned}\lambda_k(D) &\geqslant p_i^{\text{in}} + \lambda_b(H_i) + (|W_k| - \lambda_b(H_i\langle B_i\rangle))\\ &\geqslant p_i^{\text{in}} + |W_k| + q_i'\\ &= |W_k| + q\\ &= \lambda_k(D\langle B\rangle) + q,\end{aligned}$$

which completes the proof. □

Corollary 11. *Let D be an extended semicomplete digraph and let q be any positive integer. Then there exists a partition, (A, B), of $V(D)$ such that $\lambda(D\langle A\rangle) \leqslant q$ and $\lambda_k(D\langle B\rangle) \leqslant \lambda_k(D) - q$, for $k = 1, 2, 3, \ldots, |V(B)|$, provided $\lambda_k(D) - q \geqslant 0$.*

Proof. If D is strong we are done, by Theorem 10, so assume that D is not strong. Let $S_1, S_2, \ldots, S_l$ be defined such that $S_i \to S_j$ when $1 \leqslant i < j \leqslant l$ and each S_i is either a strong component or a partite set in D. Let $T_i = S_1 \cup S_2 \cup \ldots \cup S_i$, for all i, and note that if $\lambda(D\langle T_{r-1}\rangle) = q$, for an $r \in \{2, 3, \ldots, l+1\}$, then $A = T_{r-1}$ and $B = V(D) - A$ is the desired partition. So assume that $\lambda(D\langle T_{r-1}\rangle) < q < \lambda(D\langle T_r\rangle)$. Then, since every path in D visits S_r at most once, S_r does not consist of isolated vertices. Hence S_r is a strong extended semicomplete digraph, so we may use Theorem 10 on S_r, with q-value equal to $q - \lambda(D\langle T_{r-1}\rangle)$, to obtain the partition (A_r, B_r) of $D\langle S_r\rangle$. Analogously to the proof of Theorem 10 we can now show that $A = T_{r-1} \cup A_r$ and $B = V(D) - A$ is the desired partition: the inequality $\lambda(D\langle A\rangle) \leqslant q$ follows from the fact that a longest path in $D\langle A\rangle$ is obtained by prepending a longest path of $D\langle T_{r-1}\rangle$ on a longest path of $D\langle A_r\rangle$. For the last inequality of the claim, let W_k be a maximum k-path subdigraph of $D\langle B\rangle$ and suppose it intersects $D\langle B_r\rangle$ in $b > 0$ paths. Then we obtain a k-path subdigraph, W_k', of D by replacing these b paths by a maximum b-path subdigraph of $D\langle S_r\rangle$ and prepending one of those paths by a longest path of $D\langle T_{r-1}\rangle$. Then $|V(W_k')| \geqslant \lambda_k(D\langle B\rangle) + q$. □

Theorem 12. *Let D be a locally in-semicomplete digraph, and let q be any positive integer. Then there exists a partition, (A, B), of $V(D)$ such that $\lambda(D\langle A\rangle) \leqslant q$ and $\lambda_k(D\langle B\rangle) \leqslant \lambda_k(D) - q$, for $k = 1, 2, 3, \ldots, |V(B)|$, provided $\lambda_k(D) - q \geqslant 0$.*

Proof. If D is strong, it is hamiltonian [3, Theorem 5.5.1]; hence, letting A consist of the first $\min\{q, |V(D)|\}$ vertices of a Hamilton path in D, we obtain the desired partition. If D is not strong, the idea is to use the method from the proof of Theorem 10. More precisely, let the H_i be the strong components of D, define p_i^{in} and p_i^{end} as in the proof of Theorem 10, and in step (c) choose the partition (A_i, B_i) of H_i such that $\lambda(H_i\langle A_i\rangle) = q_i'$ and $\lambda_k(H_i\langle B_i\rangle) = |V(H_i)| - q_i' = \lambda_k(H_i) - q_i'$ (which is possible because H_i is traceable).

Now the proof of the first inequality in the claim carries over verbatim from the proof of Theorem 10. Also, the proof of the second inequality for the case when H_i was considered in (b) holds here, using the fact that (due to the locally in-semicompleteness of D) if a vertex, $x \in H_i$, dominates a vertex, $y \in H_j$, $i \neq j$, then x dominates $V(H_j)$ (see [3, Theorem 1.10.]). Finally, suppose H_i was considered in (c) and that b paths, $P_1, P_2, \ldots, P_b$, of W_k start in H_i. Let $x_1, x_2, \ldots, x_b$ denote the terminal vertices of the paths $P_1 \cap H_i, P_2 \cap H_i, \ldots, P_b \cap H_i$ and note that, since H_i is hamiltonian or a single vertex, there is a b-path factor in $D\langle H_i\rangle$ whose paths have exactly the terminal vertices $x_1, x_2, \ldots, x_b$. Substituting these paths for the paths $P_1 \cap H_i, P_2 \cap H_i, \ldots, P_b \cap H_i$ and prepending one of them with a path of order p_i^{in}, we get the desired inequality as in the proof of Theorem 10. □

Corollary 13. *Conjecture* 2 *holds for quasi-transitive, extended semicomplete, and locally in-semicomplete digraphs.*

Proof. This is immediate from Theorem 10, Corollary 11, and Theorem 12, letting $k = 1$. □

4.2. Proof of Conjecture 3 for locally in-semicomplete and extended semicomplete digraphs

Theorem 14. *Let D be a locally in-semicomplete digraph. For every choice of positive integers, λ_1, λ_2, such that $\lambda(D) = \lambda_1 + \lambda_2$, there exists a partition of D into two digraphs, D_1 and D_2, such that $\lambda(D_i) = \lambda_i$, for $i = 1, 2$.*

Proof. As noted in the proof of Theorem 12 the claim is trivial if D is strong. Otherwise, let $q=\lambda_1$ and apply the same partition method as we did in that proof to obtain a partition, (A, B), of $V(D)$. Then we know that $\lambda(D\langle A\rangle)\leqslant q$ and (taking $k=1$) $\lambda(D\langle B\rangle)\leqslant \lambda(D)-q$, so we only need to exhibit a path of order exactly q (resp. $\lambda(D)-q$) in A (resp. B). Consider a longest path, P, in D and suppose without loss of generality that P passes through strong components $H_1, H_2, \ldots, H_t$ in that order. Let w_i denote the terminal vertex on the path $P\cap H_i$, for $i\in\{1,2,\ldots,t\}$, and denote by $p(v)$ the order of a longest path in D with terminal vertex v.

First, suppose that some H_i ($i\in\{1,2,\ldots,t\}$) is considered in part (c) (cf. the proof of Theorem 10). Using the aforementioned fact that if a vertex, $x\in H_l$, dominates a vertex, $y\in H_j$, $l\neq j$, then x dominates $V(H_j)$ as well as the fact that every non-trivial strong component is hamiltonian, we see that (if $i>1$) $p_{i-1}^{\text{end}}=p(w_{i-1})$. Thus, $p_{i-1}^{\text{end}}=p(w_{i-1})\leqslant p_i^{\text{in}}<q$, so $H_{i-1}\subseteq A$. Furthermore, by the maximality of P, we have (if $i<t$) $p_{i+1}^{\text{in}}=p(w_i)$ and, as before, $p(w_i)=p_i^{\text{end}}$. Hence $p_{i+1}^{\text{in}}=p(w_i)=p_i^{\text{end}}>q$, so $H_{i+1}\subseteq B$.

Now suppose that, for some $i\in\{2,3,\ldots,t\}$, $H_i\subseteq A$. That is, $p_i^{\text{end}}\leqslant q$, and we get $p_{i-1}^{\text{end}}=p(w_{i-1})=p_i^{\text{in}}<p_i^{\text{end}}\leqslant q$. Hence $H_{i-1}\subseteq A$. Similarly, if $H_j\subseteq B$, for some $j\in\{1,2,\ldots,t-1\}$, then $p_j^{\text{in}}\geqslant q$ and we get $p_{j+1}^{\text{in}}=p(w_j)>p_j^{\text{in}}\geqslant q$, so $H_{j+1}\subseteq B$.

Therefore, if no H_i is considered in (c), there is an $i\in\{1,2,\ldots,t-1\}$ such that $H_1\cup H_2\cup\cdots\cup H_i\subseteq A$ and $H_{i+1}\cup H_{i+2}\cup\cdots\cup H_t\subseteq B$, so the path $P\cap A$ has order $|V(P\cap A)|=p_{i+1}^{\text{in}}\geqslant q$. Thus, $|V(P\cap A)|=q$ and $|V(P\cap B)|=|V(P)|-q=\lambda(D)-q$. On the other hand, if H_i is considered in (c), for some $i\in\{1,2,\ldots,t\}$, then $H_1\cup H_2\cup\cdots\cup H_{i-1}\subseteq A$ and $H_{i+1}\cup H_{i+2}\cup\cdots\cup H_t\subseteq B$ and we can choose the partition, (A_i, B_i), of H_i in such a way that $D\langle A_i\rangle$ has a Hamilton path, P_A, and $D\langle B_i\rangle$ has a Hamilton path, P_B, with terminal vertex w_i. Now, if we append P_A to the path $P\cap(H_1\cup H_2\cup\cdots\cup H_{i-1})$, we get a path of order $p_i^{\text{in}}+q_i'=q$ and, if we prepend P_B to the path $P\cap(H_{i+1}\cup H_{i+2}\cup\cdots\cup H_t)$, we get a path of order $|V(P)|-q=\lambda(D)-q$. Hence $D_1=D\langle A\rangle$ and $D_2=D\langle B\rangle$ is the desired partition. $\square$

We now turn our attention to extended semicomplete digraphs. Recall the definition of the numbers $l_{i,k}$ in Lemma 8. In what follows we shall denote $l_{i,1}$ simply by l_i. Also, $D=S[E_1, E_2, \ldots, E_s]$ will denote an extended semicomplete digraph.

The following observation is obvious:

Proposition 15. *If one can destroy all non-trivial (E_i, E_i)-paths by removing k vertices of D, then no path of D contains more that $k+1$ vertices from E_i.*

Lemma 16. *For each $i=1,2,\ldots,s$ such that $l_i<|E_i|$, there exists a set, Z_i, of size l_i-1 such that $D-Z_i$ has no non-trivial path connecting two (possibly equal) vertices of E_i.*

Proof. Suppose $l_i<|E_i|$ and yet we cannot destroy all (E_i, E_i) paths containing at least one arc by removing some set of l_i-1 vertices. Then it follows from Menger's theorem (see [3, Theorem 7.3.1(b)]) that D contains l_i internally disjoint (E_i, E_i) paths. Since all vertices of E_i are similar, these paths can be collected to a cycle, C, containing l_i vertices of E_i. Now let $x\in E_i\cap V(C)$ be arbitrary and let z be the predecessor of x on C. As $l_i<|E_i|$, there is some vertex $y\in E_i$ which is not on C and, since x and y are similar, we have $z\to y$. Hence $C[x,z]y$ is a path covering l_i+1 vertices of E_i, contradicting the definition of l_i. $\square$

We shall call a subset, Z_i, whose removal destroys all non-trivial (E_i, E_i)-paths, an *E_i-separator*.

Lemma 17. *Let P be a longest path in D. Then, for every $i=1,2,\ldots,s$ such that $l_i<|E_i|$, the set $V(P)-V(E_i)$ contains an E_i-separator of size l_i-1.*

Proof. By Lemma 8, P contains l_i vertices from E_i for every $i=1,2,\ldots,s$. Thus, every E_i-separator must intersect P in at least l_i-1 vertices, implying that P contains every E_i-separator of size l_i-1 for those i such that $l_i<|E_i|$. $\square$

Theorem 18. *Let D be an extended semicomplete digraph. For every choice of positive integers, λ_1, λ_2, such that $\lambda(D)=\lambda_1+\lambda_2$, there exists a partition of D into two digraphs, D_1 and D_2, such that $\lambda(D_i)=\lambda_i$, for $i=1,2$.*

Proof. If D is traceable the claim is trivial: simply take a hamiltonian path, P, and let D_1 consist of the first λ_1 vertices of P and D_2 the remaining vertices. So we may assume that D is not traceable. Below we show that it is still possible to obtain the desired partition in such a way that D_1 contains the first λ_1 vertices of some longest path, P, and D_2 the remaining vertices of P.

Fix a longest path, P, of D and let (H_1, H_2) be a partition of $D\langle V(P)\rangle$ such that H_1 is the extended semicomplete digraph induced by the first λ_1 vertices of $V(P)$ and H_2 is the extended semicomplete digraph induced by the last λ_2 vertices of $V(P)$. For $i=1,2$, let P_i be the hamiltonian path of H_i induced by P.

For each $i=1,2,\ldots,s$ and $j=1,2$, denote by $E_{i,j}$ the set of vertices of E_i that belong to P_j (and hence H_j) and let $n_{i,j}=|E_{i,j}|$.

Let $I\subseteq\{1,2,\ldots,s\}$ be the set of those indices, i, such that P does not cover all of E_i. By Lemma 17, $V(P)$ contains an E_i separator, Z_i, of size l_i-1 for each $i\in I$. Let $I_1=\{i\in I : |Z_i\cap V(P_1)|\leqslant n_{i,1}-1\}$ and let $I_2=I-I_1$. Note that $|Z_i\cap V(P_2)|\leqslant n_{i,2}-1$ for all $i\in I_2$, as otherwise $|Z_i|=|Z_i\cap V(P_1)|+|Z_i\cap V(P_2)|\geqslant n_{i,1}+n_{i,2}=l_i$.

Let $U_1=\bigcup_{i\in I_1}(V(E_i)-V(P))$ and $U_2=\bigcup_{i\in I_2}(V(E_i)-V(P))$ and let $H_i'=D\langle V(P_i)\cup U_i\rangle$, for $i=1,2$. Then (H_1', H_2') is a partition of D. Let $E_{i,j}'=E_i\cap V(H_j')$, for $i=1,2,\ldots,s$ and $j=1,2$. We claim that $\lambda(H_i')=\lambda_i$, for $i=1,2$. First note that, for every $i\in I_1$, there is an $E_{i,1}'$-separator of size at most $n_{i,1}-1$ in H_1' (namely $Z_i\cap V(P_1)$) and $|E_{i,2}'|=n_{i,2}$. Analogously, for every $i\in I_2$, there is an $E_{i,2}'$-separator of size at most $n_{i,2}-1$ in H_2' (namely $Z_i\cap V(P_2)$) and $|E_{i,1}'|=n_{i,1}$.

Thus, by Proposition 15, for every $i\in\{1,2,\ldots,s\}$ and $j=1,2$, no path in H_j' covers more than $n_{i,j}$ vertices of $E_{i,j}'$. This implies that P_j (which covers precisely $n_{i,j}$ vertices of $E_{i,j}'$) is a longest path in H_j' and the proof is complete. $\square$

5. Further remarks

When trying to obtain a partition, (D_1, D_2), of a digraph, D, such that $\lambda(D_i)=\lambda_i$, for $i=1,2$ and $\lambda_1+\lambda_2=\lambda(D)$, one approach that seems natural is to put the first λ_1 vertices of some longest path, P, of D in D_1, put the rest of P in D_2, and then try to distribute the remaining vertices of D among D_1 and D_2 in an appropriate way. As we saw in the proof of Theorem 18, this approach works for extended semicomplete digraphs. It does not, however, work in general, as shown by the example in Fig. 1. Here we have a bipartite digraph, D, with $\lambda(D)=16=|V(D)|-1$. If the longest

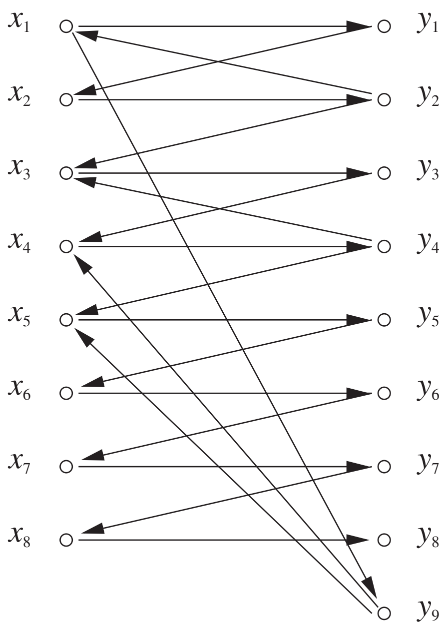

Fig. 1. A bipartite digraph for which the splitting approach does not work.

path, $P = x_1y_1 \dots x_8y_8$, is split into $P_1 = x_1y_1 \dots x_4y_4$, of order $\lambda_1 = 8$, and $P_2 = x_5y_5 \dots x_8y_8$, of order $\lambda_2 = 8$, then we cannot add the remaining vertex, y_9, to either of the digraphs induced by $V(P_i)$, $i = 1, 2$, without increasing the order of a longest path in that subdigraph.

In the introduction we pointed out that Conjecture 1 is true for undirected graphs, by choosing X as any maximal independent set. For directed graphs, a similar argument holds if we use a different notion of independence, namely *acyclic independence* (see e.g. [3, Section 12.6]). An *acyclic independent set* in a digraph, D, is a subset, $S \subseteq V(D)$, of the vertices of D, such that S induces an acyclic subdigraph of D. If we replace the term *independent set* by *acyclic independent set* in Conjecture 1, then the conjecture is true, by letting X be a maximal acyclic independent set in D.

Let $c(D)$ denote the length of a longest cycle in the digraph, D; if D is acyclic, we define $c(D)$ to be zero. We can obtain a weakened version of Conjecture 2 by relaxing the requirements $\lambda(D_i) \leqslant \lambda_i$ to $c(D_i) \leqslant \lambda_i$.

Proposition 19. *For every digraph, D, and every choice of positive integers, λ_1, λ_2, such that $\lambda(D) = \lambda_1 + \lambda_2$, there exists a partition of D into two digraphs, D_1 and D_2, such that $c(D_i) \leqslant \lambda_i$, for $i = 1, 2$.*

Proof. Let V_1 be a maximal subset of $V(D)$ satisfying that $c(D\langle V_1\rangle) \leqslant \lambda_1$, let $V_2 = V(D) - V_1$, and define $D_i = D\langle V_i\rangle$, for $i = 1, 2$. If D_2 contains a cycle, we let $C_2 = y_1y_2 \dots y_ly_1$ be a longest cycle in D_2. By maximality of V_1, $D_1 \cup \{y_l\}$ has a cycle, $C_1 = y_lx_1x_2 \dots x_{k-1}y_l$, of length $k > \lambda_1$ and, considering the path $P = y_1y_2 \dots y_lx_1 \dots x_{k-1}$,we get $\lambda_1 + \lambda_2 = \lambda(D) \geqslant |V(P)| = |V(C_2)| + k - 1 \geqslant |V(C_2)| + \lambda_1$. Thus, $c(D_2) = |V(C_2)| \leqslant \lambda_2$. □

We remark that the proofs in Section 4.1 can be turned into polynomial algorithms for finding the desired partitions. Essentially, we need to compute the numbers p_i^{in} and p_i^{end} for the strong components of non-strong quasi-transitive and locally in-semicomplete digraphs as well as find longest paths in quasi-transitive and extended semicomplete digraphs. The computation of p_i^{in} and p_i^{end} can be done using standard methods, as the strong component digraph is acyclic. A longest path in a quasi-transitive digraph can be found in $\mathcal{O}(n^5)$ (see [3, Corollary 5.10.3]) and for extended semicomplete digraphs it can be done using the result in Theorem 6.

For several classes of digraphs the following nice property holds: Given a connected subdigraph of D consisting of one path, P, and some cycles, $C_1, \dots, C_r$, all disjoint from each other and from P, there exists a path, P', in D such that $V(P') = V(P) \cup V(C_1) \cup \dots \cup V(C_r)$. This property holds for the class of semicomplete multipartite digraphs, i.e. digraphs for which the underlying undirected graph is complete multipartite (see [3, Theorem 5.7.1]). Thus, if true, the following conjecture would imply the truth of Conjecture 2 for the class of semicomplete multipartite digraphs.

Conjecture 20. *Let k be the maximum number of vertices in D that can be covered by a 1-path-cycle subdigraph of D. Then, for every choice of positive integers, k_1, k_2, such that $k = k_1 + k_2$, there exists a partition of D into two digraphs, D_1, D_2, such that no 1-path-cycle subdigraph of D_i covers more than k_i vertices of D_i, for $i = 1, 2$.*

Note that, for any digraph D, using flows in networks, one can find a 1-path-cycle subdigraph covering the maximum number of vertices from D among all such subdigraphs (see e.g. [3, Section 3.11]). Hence one can check, in polynomial time, whether a given partition of D satisfies the condition in Conjecture 20.

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